Proprietá di approssimazione di funzioni slice regolari

Irene Sabadini

Politecnico di Milano, Italy

Firenze, 23 Gennaio 2015

Preliminary results

Notation

$\mathbb{H}$: algebra of real quaternions
$q = x_0 + ix_1 + jx_2 + kx_3$
$\mathbb{S}$ denotes the set of unit purely imaginary quaternions:

$$\mathbb{S} = \{p = x_1 i + x_2 j + x_3 k \ : \ x_1^2 + x_2^2 + x_3^2 = 1\}$$

Preliminary results

Given an open set D contained in $\mathbb{C} \cong \mathbb{R}^2$ we can construct an axially symmetric open set Ω_D as

$$\Omega_D = \{q = x + Iy \mid z = x + iy \in D, \ I \in \mathbb{S}\}.$$

Definition

Let $D \subseteq \mathbb{C}$ be an open set and let $z = x + iy$ be an element in D. Let α, $\beta : D \to \mathbb{H}$ be such that $\alpha(\bar{z}) = \alpha(z)$ and $\beta(\bar{z}) = -\beta(z)$ whenever $z, \bar{z} \in D$. Let $f : \ \Omega_D \subseteq \mathbb{H} \to \mathbb{H}$ be defined by:

$$f(q) = f(x + Iy) := \alpha(x, y) + I\beta(x, y) \qquad (1)$$

for any $q = x + Iy \in \Omega_D$. Then f is said to be a *slice* function.

Preliminary results

Definition

If $\alpha, \beta : D \to \mathbb{H}$ are $\mathcal{C}^1$ and satisfy the Cauchy-Riemann system:

$$\begin{cases} \dfrac{\partial \alpha}{\partial x} - \dfrac{\partial \beta}{\partial y} = 0 \\[2ex] \dfrac{\partial \alpha}{\partial y} + \dfrac{\partial \beta}{\partial x} = 0 \end{cases}$$

the function f is said to be a (left) *slice regular* function.

Preliminary results

Definition

$$\mathcal{N}(U) = \{f \text{ slice regular in } U \ : \ f(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I, \ \ \forall I \in \mathbb{S}\}.$$

Remark

$f(x + Iy) = \alpha(x, y) + I\beta(x, y) \in \mathcal{N}(U)$ if and only if α, β are real valued.
Functions in the class $\mathcal{N}$ are called *quaternionic intrinsic* (or *radially holomorphic*) since $f(\bar{q}) = \overline{f(q)}$.
All transcendental functions belongs to the class $\mathcal{N}$.
If $f \in \mathcal{N}(U)$, $g \in \mathcal{R}(V)$ and $f(U) \subseteq V$ then the composition $g(f(q))$ is slice regular.

Preliminary results

Theorem (Runge theorem)

Let K be an axially symmetric compact set, such that $\overline{\mathbb{C}_I} \setminus (K \bigcap \mathbb{C}_I)$ is connected for some (and thus for every) $I \in \mathbb{S}$, and let Ω be an axially symmetric open set with $K \subset \Omega$. For any $f \in \mathcal{R}(\Omega)$, there exists a sequence of polynomials $P_n(q)$, $n \in \mathbb{N}$, such that $P_n \to f$ uniformly on K, as $n \to \infty$.

Riemann mapping

Mergelyan theorem

Let K be a compact subset of the complex plane $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. Then, every continuous function on K, $f : K \to \mathbb{C}$, which is holomorphic in the interior of K, can be approximated uniformly on K by polynomials.

The proof is based on Riemann mapping theorem.

Riemann mapping

Riemann Mapping Theorem (over $\mathbb{C}$)

Let $G \subset \mathbb{C}$ be a simply connected domain, $z_0 \in G$ and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk. Then there exists a unique bijective analytic function $f : G \to \mathbb{D}$ such that $f(z_0) = 0$, $f'(z_0) > 0$.

Definition

Functions defined on the open unit disc $\mathbb{D}$ which are univalent and take real values just on the real line are called typically real.

Riemann mapping

Theorem

Let $G \subset \mathbb{C}$, G nonempty, be a simply connected domain such that $G \cap \mathbb{R} \neq \emptyset$. For a fixed $x_0 \in G \cap \mathbb{R}$, there exists a unique bijective, analytic function $f: G \to \mathbb{D}$ with $f(x_0) = 0$, $f'(x_0) > 0$. The following are equivalent:

(1) f^{-1} is typically real

(2) G is symmetric with respect to the real axis

(3) f is complex intrinsic.

Riemann mapping

Remark

Let $G \subset \mathbb{C}$ be symmetric with respect to the real axis so the function f as in the statement of the Riemann mapping theorem be intrinsic, i.e. $\overline{f(\bar{z})} = f(z)$. By identifying $\mathbb{C}$ with $\mathbb{C}_J$ for some $J \in \mathbb{S}$ and using the extension formula, we obtain a function $\mathrm{ext}(f)(q)$ which is quaternionic intrinsic. Thus $\mathrm{ext}(f) \in \mathcal{N}(\Omega_G)$ where $\Omega_G \subset \mathbb{H}$ denotes the axially symmetric completion of G

Riemann mapping

Definition

Denote by $\mathfrak{R}(\mathbb{H})$ the class of axially symmetric open sets Ω in $\mathbb{H}$ such that $\Omega \cap \mathbb{C}_I$ is simply connected for every $I \in \mathbb{S}$.

Note that $\Omega \cap \mathbb{C}_I$ is simply connected for every $I \in \mathbb{S}$ and thus it is connected, so Ω is an s-domain.

Theorem

Let $\Omega \in \mathfrak{R}(\mathbb{H})$, $\mathbb{B} \subset \mathbb{H}$ be the open unit ball and let $x_0 \in \Omega \cap \mathbb{R}$. There exists a unique quaternionic intrinsic slice regular function $f : \ \Omega \to \mathbb{B}$ which is bijective and such that $f(x_0) = 0$, $f'(x_0) > 0$.

Mergelyan-type approximation results on compact sets

Theorem

Let $\Sigma \subset \mathbb{H}$ be a bounded region which is starlike with respect to the origin, and such that $\overline{\Sigma}$ is axially symmetric, and $\overline{\mathbb{C}_I} \setminus (\overline{\Sigma} \bigcap \mathbb{C}_I)$ is connected for some $I \in \mathbb{S}$. If a function f is slice regular in Σ and continuous in $\overline{\Sigma}$ then the function f can be uniformly approximated in $\overline{\Sigma}$ by polynomials.

Mergelyan-type approximation results on compact sets

Theorem

Let $T \in \mathfrak{R}(\mathbb{H})$ be bounded and such that $T \cap \mathbb{C}_I$ is a Jordan region in the plane $\mathbb{C}_I$, for all $I \in \mathbb{S}$. If f is slice regular in T, continuous in $\overline{T}$, then in $\overline{T}$ the function $f(q)$ can be uniformly approximated by polynomials in q.

Corollary

Let $T \in \mathfrak{R}(\mathbb{H})$ be bounded and such that $T \cap \mathbb{C}_I$ is a Jordan region in the plane $\mathbb{C}_I$. Let $\alpha \in \overline{T}$. If a function f is slice regular in T and continuous in $\overline{T}$ then f can be uniformly approximated in $\overline{T}$ by polynomials which take the value $f(\alpha)$ at the point α.

Mergelyan-type approximation results on compact sets

Theorem

Let $T \in \mathfrak{R}(\mathbb{H})$ be such that $T \cap \mathbb{C}_I$ is a bounded Jordan region for all $I \in \mathbb{S}$. Let $\{T_n\}$ be a sequence in $\mathfrak{R}(\mathbb{H})$ such that $\overline{T} \subset T_n$, $\overline{T}_{n+1} \subset T_n$, for all $n = 1, 2, \ldots$ and no point exterior to T belongs to T_n for all n. Let us assume that $q = 0$ belongs to T and that the bijective, quaternionic intrinsic functions $\Phi: \ T \to \mathbb{B}$, $\Phi_n: \ T_n \to \mathbb{B}$ all map $q = 0$ to $w = 0$ and $\Phi'(0) > 0$, $\Phi'_n(0) > 0$ for all $n = 1, 2, \ldots$. Then:

$$\lim_{n \to \infty} \Phi_n(q) = \Phi(q) \qquad (2)$$

uniformly for $q \in \overline{T}$.

Mergelyan-type approximation results on compact sets

The proofs of the following approximation results are interesting from two points of views:

a) they are completely constructive being based on convolutions with a well-known trigonometric kernel;

b) they allow, in addition, to obtain quantitative estimate in terms of the modulus of continuity (fact which does not happen in the previous approximation results).

Mergelyan-type approximation results on compact sets

Definition

The de la Vallée Poussin kernel is given by

$$K_n(u) = \frac{(n!)^2}{(2n)!}\left(2\cos\frac{u}{2}\right)^{2n} = 1 + 2\sum_{j=1}^{n}\frac{(n!)^2}{(n-j)!(n+j)!}\cos(ju),\ u \in \mathbb{R}.$$

Mergelyan-type approximation results on compact sets

Let $f(q)=\sum_{k=0}^{\infty} q^k c_k$, $q\in B(0;R)$, and let f be continuous on $\overline{B(0;R)}$. Define the convolution operator of quaternion variable

$$T_{n,I}(f)(q)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(qe^{I_q u})K_n(u)du,\ q\in\mathbb{H}\setminus\mathbb{R},\quad q=re^{I_q t}\in\overline{B(0;R)}$$

$$T_{n,I}(f)(q)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(qe^{Iu})K_n(u)du,$$

$$q\in\mathbb{R}\setminus\{0\},\ q=|q|e^{It}\in\overline{B(0;R)}, t=0 \text{ or } \pi,\ I\in\mathbb{S}$$

and

$$T_{n,I}(f)(0)=\frac{1}{2\pi}f(0)\int_{-\pi}^{\pi}K_n(u)du=f(0).$$

Mergelyan-type approximation results on compact sets

Theorem (Quantitative approximation on compact balls)

Let $R > 0$, $B(0;R) = \{q \in \mathbb{H}; |q| < R\}$, $K = \overline{B(0;R)}$ and $f : K \to \mathbb{H}$ be continuous on K and slice regular on the interior of K, $int(K) = B(0;R)$. Then for any $\varepsilon > 0$ there exists a polynomial P such that $|f(q) - P(q)| < \varepsilon$ for all $q \in K$.
In fact, for all $q \in \overline{B(0;R)}$ and $n \in \mathbb{N}$ we have

$$|P_n(q) - f(q)| \leq 3(R+1)\omega_1(f;1/\sqrt{n}),$$

where $\omega_1(f;\delta) = \sup\{|f(u) - f(v)|; u, v \in \overline{B(0;R)}, |u - v| \leq \delta\}$ and $P_n(q)$ are the polynomials given by

$$P_n(q) = T_{n,l}(f)(q) = \sum_{l=0}^{n} q^l c_l \cdot \frac{(n!)^2}{(n-l)!(n+l)!}, \quad q \in B(0,R)$$

where $f(q) = \sum_{k=0}^{\infty} q^k c_k$ for $q \in B(0;R)$.

Mergelyan-type approximation results on compact sets

For $q_0 = x_0 + Iy_0 \in \mathbb{H}$, with $x_0, y_0 \in \mathbb{R}$, $y_0 > 0$, $I \in \mathbb{S}$ and $R > 0$, let us denote

$$B(x_0 + y_0\mathbb{S}; R) = \{q \in \mathbb{H}; |(q - x_0)^2 + y_0^2| < R^2\}.$$

A set of the form $B(x_0 + y_0\mathbb{S}; R)$ will be called Cassini cell (it is defined by means of the so-called Cassini pseudo-metric). If $y_0 = 0$, then clearly we have $B(x_0 + \mathbb{S}y_0; R) = B(x_0; R)$.

Mergelyan-type approximation results on compact sets

Theorem (Quantitative approximation on Cassini cells)

Let $q_0 = x_0 + Iy_0 \in \mathbb{H}$, with $x_0, y_0 \in \mathbb{R}$, $y_0 > 0$, $I \in \mathbb{S}$ and $R > 0$. If $f : \overline{B(x_0 + y_0\mathbb{S}; R)} \to \mathbb{H}$ is continuous in $\overline{B(x_0 + y_0\mathbb{S}; R)}$ and slice regular in $B(x_0 + y_0\mathbb{S}; R)$, then for any $\varepsilon > 0$ there exists a polynomial P such that $|f(q) - P(q)| < \varepsilon$ for all $q \in \overline{B(x_0 + y_0\mathbb{S}; R)}$.
In fact, for all $q \in \overline{B(x_0 + y_0\mathbb{S}; R)}$ and $n \in \mathbb{N}$ we have

$$|V_n(f)(q) - f(q)| \leq 3(R+1)\omega_1(f; 1/\sqrt{n}),$$

where $V_n(f)(q)$ is a suitable polynomial.

Mergelyan-type approximation results on compact sets

Theorem

Let $\partial\Omega$ be the boundary of an open set belonging to $\mathfrak{R}(\mathbb{H})$ and containing the origin and let us assume that $\partial\Omega \cap \mathbb{C}_I$ is a Jordan curve for any $I \in \mathbb{S}$. Then a continuous slice function f on $\partial\Omega$ can be uniformly approximated on $\partial\Omega$ by polynomials in q and q^{-1}.

Universality properties

Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and
$\mathcal{M}_R = \{K; K \text{ compact }, K \subset (\mathbb{C} \setminus \overline{\mathbb{D}}_R) \text{ with } \mathbb{C} \setminus K \text{ connected }\}$.
Then $g(z) = \sum_{k=0}^{\infty} a_k z^k$ converging on $\mathbb{D}_R$ has the *universal approximation property*, if for any $K \in \mathcal{M}_R$ and any $f : K \to \mathbb{C}$, continuous on K and analytic on the interior of K, there exists a subsequence $(S_{n_k}(z))_{k\in\mathbb{N}}$ of the partial sums sequence of g such that $\lim_{k\to\infty} S_{n_k}(z) = f(z)$ uniformly for $z \in K$.

Universality properties

Definition

We say that the phenomenon of *almost universal power series* holds, if there exists a power series of radius of convergence R, with the property that for any open set Ω such that $\Omega \bigcap \overline{\mathbb{D}}_R = \emptyset$, and any compact set $K \subset \Omega$ such that $\mathbb{C} \setminus K$ is connected, given any function $f : \Omega \to \mathbb{C}$ analytic in Ω, there exists a subsequence of the partial sums sequence which converges uniformly to f on K.

Universality properties

Definition

Let $\Omega \subseteq \mathbb{H}$ be an open set and let $\mathcal{F}(\Omega)$ be the set of all axially symmetric compact sets $K \subset \Omega$, such that $\mathbb{C}_I \setminus (K \cap \mathbb{C}_I)$ is connected for some (and hence for all) $I \in \mathbb{S}$.

Universality properties

Theorem

There exists a quaternionic power series $S(q) = \sum_{k=0}^{\infty} q^k a_k$*, with radius of convergence* 1 *such that, denoting by* $S_n(q)$ *the n-th partial sum* $\sum_{k=0}^{n} q^k a_k$ *of* S*, it holds: for every* $K \in \mathcal{F}(\mathbb{H} \setminus \overline{B(0;1)})$*, for every axially symmetric open subset* Ω *of* $\mathbb{H}$ *containing* K *and for every* $f \in \mathcal{R}(\Omega)$*, there exists a subsequence* $(S_{n_k}(q))_{k\in\mathbb{N}}$ *of the partial sums of* S *such that* $S_{n_k}(q) \to f(q)$ *uniformly on* K*, as* $k \to \infty$*.*

Universality properties

Lemma

Let $K \in \mathcal{F}(\mathbb{H} \setminus \overline{B(0;1)})$ and let Ω be an axially symmetric open subset of $\mathbb{H}$ containing K. Then, every $f \in \mathcal{R}(\Omega)$ can be uniformly approximated on K by polynomials $p_n(q) = a_{n,0} + qa_{n,1} + \ldots + q^n a_{n,n}$ such that $\lim_{n\to\infty} \sum_{k=0}^{n} |a_{n,k}|^2 = 0$. In particular, every $f \in \mathcal{R}(\Omega)$ can be uniformly approximated on K by polynomials $p_n(q) = a_{n,0} + qa_{n,1} + \ldots + q^n a_{n,n}$ whose coefficients $a_{n,j}$ satisfy $|a_{n,j}| \leq 1$, $j = 0, 1, \ldots, n$.

Universality properties

Theorem (Birkhoff-Seidel-Walsh)

There exists an entire function $F(z)$ such that given an arbitrary function $f(z)$ analytic in a simply connected region $R \subseteq \mathbb{C}$, for suitably chosen $a_1, a_2, \ldots$ the relation

$$\lim_{n \to \infty} F(z + a_n) = f(z)$$

holds for $z \in R$ uniformly on any compact set in R.

Universality properties

Theorem

There exists a quaternionic entire function $F(q)$ such that given an arbitrary function $f(q)$ slice regular in a region $\Omega \in \mathfrak{R}(\mathbb{H})$, for suitably chosen $a_1, a_2, \ldots \in \mathbb{R}$ the relation

$$\lim_{n\to\infty} F(q + a_n) = f(q) \tag{3}$$

holds for $q \in \Omega$ uniformly on any compact set in $\mathcal{F}(\mathbb{H})$ contained in Ω.

Universality properties

Theorem

There exists an entire slice regular function F such that the set $\{F^{(n)}\}_{n\in\mathbb{N}}$ is dense in $\mathcal{R}(\mathbb{H})$.

Theorem

Let $E\subset\mathbb{H}$ be a (not necessarily bounded) closed, axially symmetric set with no holes and such that for any closed ball B of center at a real number, the union of all holes of $E\bigcup B$ is a bounded set. If f is a slice regular function in an neighborhood of E, then for any $\varepsilon>0$, there exists an entire slice regular function h such that $|f(q)-h(q)|<\varepsilon$, for all $q\in E$.

References

1. S.G. Gal, I. S., *Approximation by polynomials on quaternionic compact sets*, to appear in Math. Met. Appl. Sci.
2. S.G. Gal, I. S., *Universality properties of the quaternionic power series and entire functions*, to appear in Math. Nach.